

CYCLE STRUCTURE OF PERMUTATION FUNCTIONS OVER FINITE FIELDS AND THEIR APPLICATIONS

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ABSTRACT. In this work we establish some new interleavers based on permutation functions. The inverses of these interleavers are known over a finite field \mathbb{F}_q . For the first time Möbius and Rédei functions are used to give new deterministic interleavers. Furthermore we employ Skolem sequences in order to find new interleavers with known cycle structure. In the case of Rédei functions an exact formula for the inverse function is derived. The cycle structure of Rédei functions is also investigated. The self-inverse and non-self-inverse versions of these permutation functions can be used to construct new interleavers.

1. INTRODUCTION

Interleavers have a lot of impact on various aspects of Communication Theory. For example, a crucial role in designing good turbo codes and turbo lattices is played by interleavers [20, 28, 29]. The distance properties of turbo codes can change dramatically from one interleaver to another [4]. In addition, bit interleavers are employed at the encoder of a coded modulation scheme over fading channels to improve the error performance [5].

Several studies have been conducted on deterministic interleavers. One of the main advantages of well-known deterministic interleavers is that they have a simple structure that is easy to implement. Only some defining parameters of the interleaver, such as the coefficients of a polynomial, are stored. Efforts in the field of deterministic interleavers have focused on permutation polynomials [30, 31]. Therefore, it seems natural to search in the class of permutation functions for finding good interleavers. Recent works concerning the inverse of interleavers like [8, 22, 26, 32] motivated us to turn our attention to permutations which their inverses are given. Interleavers with known inverses are of interest [8, 22] because the same structure and technology used for encoding can be used for decoding as well. The permutation polynomials used in these deterministic interleavers are all over the integer ring \mathbb{Z}_n .

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Providing interleavers based on permutation functions is the main contribution of this work. In this paper we use permutation functions over \mathbb{F}_q and Skolem sequences to create interleavers with known inverses. Using results from finite fields, we can construct permutation polynomials and permutation functions over \mathbb{F}_q . However, only very special cases of permutation functions are known. Some examples are permutation monomials, Dickson permutation polynomials, nonlinear transformation (Möbius) and Rédei permutation functions.

The study of permutation monomials x^n with a cycle of length j has been treated in [1]. Permutation monomials x^n with all cycles of the same length are characterized in [24]. The cycle structure of Dickson permutation polynomials $D_n(x, a)$ where $a \in \{0, \pm 1\}$ has been studied in [17]. The cycle structure of Möbius transformation has been described in [7]. In this paper, we provide an exact formula for the inverse of every Rédei function and study the cycle structure of Rédei functions. More precisely, Rédei functions with a cycle of length j are characterized. Then we extend this to all cycles of the same length j or 1. An exact formula for the number of cycles of length j is given. These are other contributions of this study.

Skolem sequences have been introduced and studied extensively [2, 16]. These sequences have applications in constructing cyclic Steiner triple systems and in constructing codes resistant to random interference [9, 14]. In this work we continue to find applications for these nice structured sequences. Specially, we use these sequences to produce self-inverse and non-self-inverse interleavers.

This paper is organized as follows: For making the paper self-contained, background on interleavers and on permutation functions are given in Section 2. The general structure of our deterministic interleavers is explained and investigated in this section. Monomial, Dickson, Möbius, Rédei and Skolem interleavers are studied in Section 3. The cycle structure of Rédei functions as well as the number of cycles of certain length are also given in this section. Conclusions and further work are commented in Section 4.

2. BACKGROUND

2.1. BASIC DEFINITION OF INTERLEAVERS.

Let us first give the general concept of an interleaver. An interleaver Π may be interpreted as a function which permutes the indices of components of \mathbf{u} . In other words, let $I = \{0, 1, \dots, N-1\}$ be all indices of a vector $\mathbf{u} = (u_0, \dots, u_{N-1})$, then the interleaver Π can be considered as a bijective function of I . The inverse function Π^{-1} is also necessary for decoding process when we implement a *deinterleaver*. An interleaver Π is called *self-inverse* if $\Pi = \Pi^{-1}$.

2.2. SOME WELL-KNOWN PERMUTATION FUNCTIONS OVER FINITE FIELDS.

Let $q = p^m$ and \mathbb{F}_q be the finite field of order q where p is a prime number. A *permutation function* over \mathbb{F}_q is a bijection that sends the elements of \mathbb{F}_q onto itself. It is clear that permutation functions have a functional inverse with respect to composition. Thus, for a permutation function $P \in \mathbb{F}_q[x]$, there exists a unique $P^{-1} \in \mathbb{F}_q[x]$ of degree less than q such that $P(P^{-1}(x)) = P^{-1}(P(x)) = x \pmod{x^q - x}$ for all $x \in \mathbb{F}_q$. A permutation function P is called *self-inverse* if $P = P^{-1}$.

Let f be a primitive polynomial of degree m over \mathbb{F}_p and assume that α is a root of f . Since $q = p^m$ and f divides $x^{q-1} - 1$, we can represent \mathbb{F}_q as

$$(1) \quad \mathbb{F}_q = \{0, \alpha^1, \dots, \alpha^{q-2}, \alpha^{q-1}\},$$

where $\alpha^{q-1} = 1$. The above representation (power representation) of \mathbb{F}_q is appropriate for operations like multiplication and raising to a power. Furthermore, for every α^i , $0 \leq i \leq q-1$, there exists a polynomial representation (in α) with degree less than m which is adequate for addition and subtraction.

Next, we review four well-known permutation functions on the finite field \mathbb{F}_q . They are useful for constructing new deterministic interleavers.

- Monomials [19]: $M(x) = x^n$ for some $n \in \mathbb{N}$ is a permutation polynomial over \mathbb{F}_q if and only if $\gcd(n, q-1) = 1$ where \gcd denotes the greatest common divisor. The inverse of $M(x)$ is obviously the monomial $M^{-1}(x) = x^m$ where $nm \equiv 1 \pmod{q-1}$.
- Dickson polynomials of the 1st kind [19]: Let n be an integer. A function D_n which satisfies $D_n(x+y, xy) = x^n + y^n$ for all $x, y \in \mathbb{F}_q$ is called a Dickson polynomial over \mathbb{F}_q . Let us fix $y = a$ for an element $a \in \mathbb{F}_q$. The function $D_n(x, a)$ is a permutation polynomial if and only if $\gcd(n, q^2-1) = 1$. For $a \in \{0, \pm 1\}$, the inverse of $D_n(x, a)$ is $D_m(x, a)$ where $nm \equiv 1 \pmod{q^2-1}$. It is easy to check [19] that

$$(2) \quad D_n(x, a) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}.$$

- Möbius transformation: The function

$$(3) \quad T(x) = \begin{cases} \frac{ax+b}{cx+d} & x \neq \frac{-d}{c}, \\ \frac{a}{c} & x = \frac{-d}{c}, \end{cases}$$

where $a, b, c, d \in \mathbb{F}_q$, $c \neq 0$ and $ad - bc \neq 0$ is a permutation function. Its inverse is simply

$$(4) \quad T^{-1}(x) = \begin{cases} \frac{dx-b}{-cx+a} & x \neq \frac{a}{c}, \\ \frac{-d}{c} & x = \frac{a}{c}. \end{cases}$$

- Rédei functions [23]: Let $\text{char}(\mathbb{F}_q) \neq 2$ and $a \in \mathbb{F}_q^*$ be a non-square element; then $\sqrt{a} \in \mathbb{F}_{q^2}^*$. The numerator $G_n(x, a)$ and the denominator $H_n(x, a)$ of the Rédei function are polynomials in $\mathbb{F}_q[x]$ satisfying the equation

$$(x + \sqrt{a})^n = G_n(x, a) + H_n(x, a)\sqrt{a}.$$

Also, it is easy to see that

$$(x - \sqrt{a})^n = G_n(x, a) - H_n(x, a)\sqrt{a}.$$

The Rédei function $R_n = \frac{G_n}{H_n}$ with degree n is a rational function over \mathbb{F}_q .

We have that R_n is a permutation function if and only if $\gcd(n, q+1) = 1$. In addition, if $\text{char}(\mathbb{F}_q) \neq 2$ and $a \in \mathbb{F}_q^*$ is a square element, then R_n is a permutation function if and only if $(n, q-1) = 1$.

2.3. SKOLEM SEQUENCES.

Let D be a set of integers. A *Skolem-type sequence* is a sequence with alphabet D where each element $i \in D$ appears exactly twice in the sequence at positions a_i and $a_i + i = b_i$. Thus, $|b_i - a_i| = i$ for every $i \in D$. These sequences might have empty positions, which we fill with zeros. For more information about Skolem sequences, we refer the reader to [16]. Here we use Skolem sequences to construct self-inverse interleavers of a specific size. A partition of the set $[n] = \{1, \dots, n\}$ into n ordered pairs $\{(a_i, b_i) : b_i - a_i = i, 1 \leq i \leq n\}$, implies a *Skolem sequence of order n* . It is

obvious that in order to generate Skolem sequence corresponding to this partition we must put integer $i \in [n]$ in positions a_i and b_i of a sequence $S = (s_1, \dots, s_{2n})$. A *k-extended Skolem sequence* of order n is a Skolem sequence of order n which contains exactly one hole in position k . If k is in the penultimate position, the sequence is called a *hooked sequence*. A (j, n) -*generalized Skolem sequence* of multiplicity j is a sequence $S = (s_1, \dots, s_t)$ of integers from $[n]$ such that for every $i \in [n]$ there are exactly j positions in the sequence S , let us say $r_1, r_2 = r_1 + i, \dots, r_j = r_1 + (j-1)i$, such that $s_{r_1} = s_{r_2} = \dots = s_{r_j} = i$. It is easy to see that $t = jn$ in this case.

Example 2.1. The sequence $(4, 1, 1, 3, 4, 2, 3, 2)$ is a Skolem sequence of order 4. The sequence $(2, 5, 2, 6, 1, 1, 5, 3, 4, 6, 3, 0, 4)$ is a hooked Skolem sequence of order 6.

In the following we mention two theorems from [16] which state necessary and sufficient conditions for the existence of the above defined Skolem sequences.

Theorem 2.2. *A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod{4}$. A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod{4}$. A k -extended Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod{4}$, when k is odd and $n \equiv 2, 3 \pmod{4}$ when k is even.*

Theorem 2.3. *Let $j = p^e t$, where p is the smallest prime factor of j , and e, t are positive integers. Then a (j, n) -generalized Skolem sequence exists if and only if $n \equiv 0, 1, \dots, p-1 \pmod{p^{e+1}}$.*

There is an efficient heuristic algorithm for obtaining a Skolem sequence of large order based on hill-climbing [15]. The sufficiency of the above existence theorems is usually proved by giving direct constructions of the required sequences. New sequences can also be found by concatenating two or more existing sequences.

Several direct constructions of hooked extended Skolem sequences are provided in [21] in terms of unions or sums of two Skolem sequences. Pivoting and doubling are another techniques for constructing Skolem sequences [16]. When $n > 5$, we can use the explicit constructions given in [9] to find our ordered pairs for Skolem and hooked Skolem sequences.

By means of various types of Skolem sequences we introduce Skolem interleavers with known cycle structure. Also we can provide interleavers with cycles of length j or 1 by using generalized Skolem sequences.

3. NEW ALGEBRAIC INTERLEAVERS OVER \mathbb{F}_q

3.1. PERMUTATION FUNCTIONS AS DETERMINISTIC INTERLEAVERS. As before, let α be a primitive element, root of a primitive polynomial f over \mathbb{F}_p of degree m , and $q = p^m$. If P is a permutation function over $\mathbb{F}_q = \{0, \alpha^1, \dots, \alpha^{q-2}, \alpha^{q-1}\}$, then for every $1 \leq i \leq q-1$ there exists a j such that $P(\alpha^i) = \alpha^j$, $1 \leq j \leq q-1$. Thus, $P^{-1}(\alpha^j) = \alpha^i$. In this manner, we can take this permutation function P as a function which rearranges the powers of α . Therefore, every permutation function over the finite field \mathbb{F}_q can induce an interleaver as follows:

Definition 3.1. Let α be a root of a primitive polynomial f over \mathbb{F}_p of degree m , and $q = p^m$. Let P be a permutation function over \mathbb{F}_q . An interleaver $\Pi_P : \mathbb{Z}_q \rightarrow \mathbb{Z}_q$ is defined by

$$(5) \quad \Pi_P(i) = \ln(P(\alpha^i))$$

where $\ln(\cdot)$ denotes the discrete logarithm to the base α over \mathbb{F}_q^* and $\ln(0) = 0$.

It is easy to see that every permutation function P has a unique compositional inverse P^{-1} . Clearly, P^{-1} is also a permutation function over \mathbb{F}_q . We can also define the interleaver $\Pi_{P^{-1}}$ by means of P^{-1} . Based on the above discussions the following statements can be described [27]. There is a one-to-one correspondence between the set of all permutation functions over a fixed finite field \mathbb{F}_q and the set of all interleavers of size q . One of the straightforward consequences of the above facts is that for a self-inverse permutation function P over \mathbb{F}_q , we have $\Pi_P = (\Pi_P)^{-1}$.

We now proceed to introduce interleavers based on permutation functions over finite fields based on the above discussions. The following general definition works for all of them.

Definition 3.2. Let f be one of the permutation functions over \mathbb{F}_q that were cited in Section 2. Then Π_f as defined in (5) is a new deterministic interleaver. Each of them can be explicitly named by their underlying permutation function.

For example, we have *monomial*, *Dickson*, *Möbius*, *Rédei* and *Skolem interleavers*. The classes of permutation functions with explicit inverse formulas are cited in Section 2. Each of them induce a new deterministic interleaver based on the above definition.

3.2. MONOMIAL INTERLEAVERS. Let $M(x) = x^n$ over \mathbb{F}_q and $\gcd(n, q-1) = 1$. Then

$$\Pi_M(i) = \ln(M(\alpha^i)) = \ln((\alpha^i)^n) = ni \pmod{q-1},$$

for $i \in \mathbb{Z}_q$. So, $\Pi_M(x) = nx \pmod{q-1}$ for $x \in \mathbb{Z}_q$. Since $\gcd(n, q-1) = 1$, Π_M is a linear permutation polynomial.

Example 3.3. Assume that $n = 11$ and $q = 13$. Since $\gcd(11, 12) = 1$, the monomial $M(x) = x^{11}$ is a permutation polynomial over \mathbb{F}_{13} . Furthermore, $11 \cdot 11 \equiv 1 \pmod{12}$ and this means that $M^{-1} = M$ is a permutation polynomial over \mathbb{F}_{13} and $\Pi_{M^{-1}} = (\Pi_M)^{-1}$. Therefore, $\Pi_{M^{-1}} = \Pi_M$ can act as the deinterleaver too. Since 2 is a primitive element of \mathbb{F}_{13} , we get

$$\begin{aligned} M(2^1) &= 2^{11}, & M(2^2) &= 2^{10}, & M(2^3) &= 2^9, & M(2^4) &= 2^8, \\ M(2^7) &= 2^5, & M(2^6) &= 2^6, & M(2^5) &= 2^7, & M(2^8) &= 2^4, \\ M(2^9) &= 2^3, & M(2^{10}) &= 2^2, & M(2^{11}) &= 2^1, & M(2^{12}) &= 2^{12}. \end{aligned}$$

We can interpret this interleaver and its deinterleaver using

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 12 \end{pmatrix}.$$

Also we observe that the only three fixed points are 0, $2^6 = -1 \pmod{13}$ and $2^{12} = 1 \pmod{13}$; this is a general fact as it will be seen in Corollary 3.5.

Our approach leads us to use self-inverse permutation functions which have cycles of the same length $j = 2$, or otherwise fixed points. Such permutation monomials are obtained using the following theorem from [24] for $j = 2$. We recall that $j = \text{ord}_s(n)$, if j is the smallest integer with the property $n^j \equiv 1 \pmod{s}$.

Theorem 3.4. Let $q-1 = p_0^{k_0} p_1^{k_1} \dots p_r^{k_r}$. The permutation monomial $M(x) = x^n$ of \mathbb{F}_q has only cycles of the same length j or 1 (fixed points) if and only if one of the following conditions holds for each $0 \leq \ell \leq r$:

- $n \equiv 1 \pmod{p_\ell^{k_\ell}}$,
- $j = \text{ord}_{p_\ell^{k_\ell}}(n)$ and $j|p_\ell - 1$,

- $j = \text{ord}_{p_\ell^{k_\ell}}(n)$, $k_\ell \geq 2$ and $j = p_\ell$.

Since we concentrate on the case $j = 2$, the following corollary is useful for us.

Corollary 3.5. *Let $q - 1 = p_0^{k_0} p_1^{k_1} \dots p_r^{k_r}$ where $p_0 = 2$. The permutation polynomial of \mathbb{F}_q given by $M(x) = x^n$ decomposes in cycles of the same length j and $\{0, 1, -1\}$ are the only fixed elements if and only if*

- for $k_0 > 2$: $j = 2$ and $n = q - 2$ or $n = \frac{q-3}{2}$.
- for $k_0 = 2$: $j = 2$ and $n = q - 2$.

We note that if we use $n = q - 2$, we get some sort of symmetry in our permutation as in Example 3.3. Let $n = q - 2$, then we have $\Pi_M(x)$ equals to

$$\ln(M(\alpha^x)) = \ln(\alpha^{x(q-2)}) = x(q-2) \pmod{q-1}.$$

It is easy to see that for every $i \in \mathbb{Z}_q$ we have $\Pi_M(i) = q - 1 - i$ and $\Pi_M(q - 1 - i) = i$ because $\Pi_M(i) = i(q - 2) = i(q - 1 - 1) = i(q - 1) - i = -i = q - 1 - i \pmod{q-1}$ and since the permutation is self-inverse, $\Pi_M(q - 1 - i) = i$. However, this is not the case for $n = \frac{q-3}{2}$ in Corollary 3.5. In this situation $\Pi_M(x) = x \left(\frac{q-3}{2} \right) \pmod{q-1}$. We do not see that symmetry in this case and it seems that these self-inverse monomial interleavers perform better than self-inverse monomial interleavers with $n = q - 2$. The authors of [10] have constructed and investigated the efficiency of self-inverse monomial interleavers with $n = q - 2$.

3.3. DICKSON INTERLEAVERS. Let $\gcd(n, q^2 - 1) = 1$. It is known [19] that $D_n(x, a)$ for $a \in \{0, \pm 1\}$ is a permutation polynomial over \mathbb{F}_q and has the compositional inverse $D_m(x, a)$ where $nm \equiv 1 \pmod{q^2 - 1}$. We can define a set of deterministic interleavers $\Pi_D^{(n,a)} : \mathbb{Z}_q \rightarrow \mathbb{Z}_q$ by $\Pi_D^{(n,a)}(i) = \ln(D_n(\alpha^i, a))$.

Example 3.6. Let $n = 19$, $q = 11$ and $a = 1$. Then we get

$$D_{19}(x, 1) = x^9 + 3x^7 + 9x^5 + 5x^3 + 5x \pmod{11}.$$

Since $\gcd(19, (11)^2 - 1) = \gcd(19, 120) = 1$ and $a = 1$, $D_{19}(x, 1)$ is a permutation polynomial over \mathbb{F}_{11} with compositional inverse $D_m(x, 1)$ where $19m \equiv 1 \pmod{120}$. Therefore, $m = 19$, and this means that $D_{19}(x, 1)$ is a self-inverse Dickson permutation polynomial over \mathbb{F}_{11} . A Dickson interleaver $\Pi_D^{(19,1)} : \mathbb{Z}_{11} \rightarrow \mathbb{Z}_{11}$ can be defined by $\Pi_D^{(19,1)}(i) = \ln(D_{19}(2^i, 1))$ where $2 \in \mathbb{F}_{11}$ is a primitive element. Thus, we have the following:

$$\begin{aligned} D_{19}(0, 1) &= 0, & D_{19}(2, 1) &= 2, & D_{19}(2^2, 1) &= 2^2, \\ D_{19}(2^3, 1) &= 2^3, & D_{19}(2^4, 1) &= 2^9, & D_{19}(2^5, 1) &= 2^5, \\ D_{19}(2^6, 1) &= 2^6, & D_{19}(2^7, 1) &= 2^7, & D_{19}(2^8, 1) &= 2^8, \\ D_{19}(2^9, 1) &= 2^4, & D_{19}(2^{10}, 1) &= 2^{10}. \end{aligned}$$

This means that $\Pi_D^{(19,1)}$ permutes the elements of \mathbb{Z}_{11} as follow

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 & 10 \end{pmatrix}.$$

The following two theorems are from [25]. We use $j = \text{ord}_s^-(n)$ for the least integer with $n^j \equiv -1 \pmod{s}$.

Theorem 3.7. Let $q - 1 = p_1^{k_1} \dots p_r^{k_r}$ and $q + 1 = p_{r+1}^{k_{r+1}} \dots p_s^{k_s}$ be the prime factorization of $q - 1$ and $q + 1$, respectively. Suppose that $\gcd(n, q^2 - 1) = 1$. The Dickson permutation polynomial $D_n(x, 1)$ over \mathbb{F}_q is the identity on \mathbb{F}_q or all the non-trivial cycles have length two if and only if one of the following holds for all $1 \leq \ell \leq r$, and one of the following conditions holds for all $r + 1 \leq \ell \leq s$:

1. Either
 - (a) $n \equiv 1 \pmod{p_\ell^{k_\ell}}$ and $p_\ell^{k_\ell} = 2$, or
 - (b) $2 = \text{ord}_{p_\ell^{k_\ell}}^-(n)$, and $4 \mid (p_\ell - 1)$.
2. Either
 - (a) $n \equiv \pm 1 \pmod{p_\ell^{k_\ell}}$, or
 - (b) $2 = \text{ord}_{p_\ell^{k_\ell}}(n)$, $p_\ell = 2$, $k_\ell \geq 2$, and $n \not\equiv -1 \pmod{p_\ell^{k_\ell}}$.

Theorem 3.8. Let $q - 1 = p_1^{k_1} \dots p_r^{k_r}$ and $q + 1 = p_{r+1}^{k_{r+1}} \dots p_s^{k_s}$ be the prime factorization of $q - 1$ and $q + 1$, respectively. Suppose that $\gcd(n, q^2 - 1) = 1$. The Dickson permutation polynomial $D_n(x, -1)$ over \mathbb{F}_q is the identity on \mathbb{F}_q or all the non-trivial cycles have length two if and only if one of the following conditions holds for all $1 \leq \ell \leq r$, and one of the following conditions holds for all $r + 1 \leq \ell \leq s$:

1. Either
 - (a) $2(n + 1) \equiv 0 \pmod{p_\ell^{k_\ell}}$ and $p_\ell^{k_\ell} = 2, 4$, or
 - (b) $2 = \text{ord}_{p_\ell^{k_\ell}}^-(n)$, and $4 \mid (p_\ell - 1)$
2. Either
 - (a) $2(n + 1) \equiv 0 \pmod{p_\ell^{k_\ell}}$, or
 - (b) $n \equiv 1 \pmod{p_\ell^{k_\ell}}$, or
 - (c) $2 = \text{ord}_{p_\ell^{k_\ell}}(n)$, $k_\ell \geq 2$ and $p_\ell = 2$.

If $D_n(x, 1)$ meets the assumptions of Theorem 3.7, then all the cycles of $D_n(x, 1)$ have the same length 1 or 2. This means that $D_n(x, 1)$ is a self-inverse permutation polynomial, that is, $D_n(x, 1) = D_m(x, 1)$ where $mn \equiv 1 \pmod{q^2 - 1}$. Hence, we conclude that $\Pi_D^{(n, 1)}$ is a self-inverse interleaver. A similar statement can be given for $D_n(x, -1)$ using Theorem 3.8.

Furthermore, by using the above theorems and other results in [25] we can derive Dickson polynomials which produce permutations with prescribed cycle structure. Hence these polynomials can be employed to construct both self-inverse and non-self-inverse interleavers. If $a = 0$ then these Dickson polynomials turn out to be monomials. Thus, their corresponding Dickson interleavers can be considered as a generalization of monomial interleavers.

3.4. MÖBIUS INTERLEAVERS. In the following we establish and introduce Möbius interleavers. This is the first usage of these non-linear transformations to construct interleavers. In order to generate Möbius interleavers with prescribed cycle arrangement, one should know the cycle structure of these functions. This is reported from [7] and used here to provide new deterministic interleavers. Since self-inverse Möbius interleavers need only three defining parameters $a = d$, b and c , they have a simple and efficient structure that is easy to implement. One can derive the inverse function of T using (4). We have $T = T^{-1}$ if and only if $a = d$, $-b = b$ and $c = -c$. Let $q = 2^n$, we get $-b = b$ and $c = -c$. Therefore, for a self-inverse Möbius function

we have

$$(6) \quad T(x) = T^{-1}(x) = \begin{cases} \frac{ax+b}{cx+a} & x \neq \frac{a}{c}, \\ \frac{a}{c} & x = \frac{a}{c}, \end{cases}$$

where $a^2 - bc \neq 0$ and $c \neq 0$. A detail example is provided in [27].

We cite next theorem from [7]. This fully describes the cycle structure of T in terms of the eigenvalues of the coefficient matrix A_T associated to T

$$(7) \quad A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial t of A_T is a quadratic polynomial.

Theorem 3.9. *Let T be the permutation defined by (3), and let t be the characteristic polynomial of the matrix A_T associated to T . Let $\alpha_1, \alpha_2 \in \mathbb{F}_{q^2}$ be the roots of t .*

1. *Suppose $t(x)$ is irreducible. If $k = \text{ord}\left(\frac{\alpha_1}{\alpha_2}\right) = \frac{q+1}{s}$, $1 \leq s < \frac{q+1}{2}$, then T has $s-1$ cycles of length k and one cycle of length $k-1$. In particular T is a full cycle if $s=1$.*
2. *Suppose $t(x)$ is reducible and $\alpha_1, \alpha_2 \in \mathbb{F}_q^*$ are roots of $t(x)$ and $\alpha_1 \neq \alpha_2$. If $k = \text{ord}\left(\frac{\alpha_1}{\alpha_2}\right) = \frac{q-1}{s}$, $s \geq 1$, then T has $s-1$ cycles of length k , one cycle of length $k-1$ and two cycles of length 1.*
3. *Suppose $t(x) = (x - \alpha_1)^2$, $\alpha_1 \in \mathbb{F}_q^*$ where $q = p^n$. Then T has $p^{n-1} - 1$ cycles of length p , one cycle of length $p-1$ and one cycle of length 1.*

It is obvious that we are interested again on permutations with cycles of length 1 and 2 only. Hence based on the above theorem we have the following theorem:

Theorem 3.10. *Let Π_T be an interleaver defined by T , and let t , A_T , α_1 and α_2 be as in Theorem 3.9. Then Π_T is a self-inverse interleaver if $\text{Tr}(A_T) = 0$.*

Proof. The three cases of the previous theorem have the following consequences:

1. We should have $2 = k = \text{ord}\left(\frac{\alpha_1}{\alpha_2}\right)$. But $k=2$ if and only if $(\alpha_1)^2 = (\alpha_2)^2$ and $\alpha_1 \neq \alpha_2$ if and only if $\alpha_1 = -\alpha_2$ if and only if $a+d = \text{Tr}(A_T) = \alpha_1 + \alpha_2 = 0$. Hence, t is irreducible and $\text{Tr}(A_T) = 0$ if and only if $k=2$ and we have $\frac{q+1}{2} - 1$ cycles of length two and one cycle of length one.
2. In a similar situation with case 1) we have that t is reducible and $\text{Tr}(A_T) = 0$ if and only if $k=2$ and we have $\frac{q-1}{2} - 1$ cycles of length two and three cycles of length one.
3. In the third case we have only cycles of length 1 and 2 when $p=2$. As we mentioned in (6) in this case we also have $\text{Tr}(A_T) = a+d=0$. So $a=d$ and this means that $\alpha_1 = a$. Thus, $a=d$ if and only if T has $2^{n-1} - 1$ cycles of length 2 and two cycles of length 1 where $q = 2^n$.

□

3.5. RÉDEI INTERLEAVERS. In this section the concept of Rédei interleavers is introduced. Again, obtaining the cycle structure of Rédei functions is essential. In the following, the arrangement of cycles of Rédei functions is described. More precisely, Rédei functions with all cycles of length $j \neq 1$ are derived. We also provide a condition under which all cycles of a Rédei function are of length j or 1. The

inverse for every Rédei function R_n is provided next. The following relations are cited from [6] for all $x, y \in \mathbb{F}_q$

$$\begin{aligned} (8) \quad & R_n(R_m)(x, a) = R_{nm}(x, a), \\ (9) \quad & R_n(x, a) = x \iff n \equiv 1 \pmod{q+1}, \\ (10) \quad & R_n\left(\frac{xy+a}{x+y}, a\right) = \frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)}. \end{aligned}$$

The next lemma is proved in [6].

Lemma 3.11. *Let r be a rational function with coefficients in \mathbb{F}_q that satisfies*

$$(11) \quad r\left(\frac{xy+a}{x+y}\right) = \frac{r(x)r(y) + a}{r(x) + r(y)}$$

where a is a fixed element of \mathbb{F}_q and x and y are two unknowns. Then, if $a \neq 0$ and $\text{char}(\mathbb{F}_q) \neq 2$, r coincides with a Rédei's function for some m (not necessarily relatively prime to $q+1$).

Comparing (11) with (10) results in the next theorem which provides the inverse of every Rédei function.

Theorem 3.12. *Let R_n for some n be a Rédei function over \mathbb{F}_q where $a \in \mathbb{F}_q^*$ is a non-square and $\gcd(n, q+1) = 1$. Then $R_n^{-1} = R_m$ for m satisfying $nm \equiv 1 \pmod{q+1}$.*

Proof. First, it is clear that R_n has a compositional inverse R_n^{-1} . Using (10) and taking $R_n^{-1}(\cdot)$ on both sides, we get

$$\frac{xy+a}{x+y} = R_n^{-1}\left(\frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)}\right).$$

Let us assume that $s = R_n(x, a)$ and $t = R_n(y, a)$. Then applying (11) to R_n^{-1} , we get

$$\frac{R_n^{-1}(R_n(x, a))R_n^{-1}(R_n(y, a)) + a}{R_n^{-1}(R_n(x, a)) + R_n^{-1}(R_n(y, a))} = R_n^{-1}\left(\frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)}\right),$$

implying that, for all $s, t \in \mathbb{F}_q$, we get

$$\frac{R_n^{-1}(s)R_n^{-1}(t) + a}{R_n^{-1}(s) + R_n^{-1}(t)} = R_n^{-1}\left(\frac{st+a}{s+t}\right).$$

Since R_n^{-1} satisfies all the conditions of Lemma 3.11, R_n^{-1} coincides with a Rédei function for some m . So, $R_n^{-1} = R_m$. Now, we use (8) and (9) to get

$$\text{id}_{\mathbb{F}_q} = R_n(R_n^{-1}) = R_n(R_m) = R_{nm} \iff nm \equiv 1 \pmod{q+1}.$$

□

We note that R_n^{-1} is a rational function because every function from \mathbb{F}_q to itself can be interpreted as a polynomial with degree less than q .

An example is given in [27].

Theorem 3.13. *Let j be a positive integer. The Rédei function $R_n(x, a)$ of \mathbb{F}_q with $\gcd(n, q+1) = 1$ has a cycle of length j if and only if $q+1$ has a divisor s such that $j = \text{ord}_s(n)$.*

Proof. Let $R_n^{(j)}(x, a)$ denote the j -th iterate of $R_n(x, a)$ under the composition operation. We get

$$\begin{aligned} R_n^{(j)}(x, a) = x &\iff R_{n^j}(x, a) = x \iff \frac{G_{n^j}(x, a)}{H_{n^j}(x, a)} = x \\ &\iff G_{n^j}(x, a) = xH_{n^j}(x, a), \end{aligned}$$

where the first equivalence is derived from (8). Furthermore, we get the following identities for $x \in \mathbb{F}_q$

$$\begin{aligned} (x + \sqrt{a})^{n^j} &= G_{n^j}(x, a) + H_{n^j}(x, a)\sqrt{a} \\ &= xH_{n^j}(x, a) + H_{n^j}(x, a)\sqrt{a} \\ &= H_{n^j}(x, a)(x + \sqrt{a}). \end{aligned}$$

Let us assume that $y = x + \sqrt{a}$, then $y^{n^j-1} = H_{n^j}(x, a) \in \mathbb{F}_q$ and $y \in \mathbb{F}_{q^2}$. Since $y^{n^j-1} \in \mathbb{F}_q$ we have that $y^{(n^j-1)(q-1)} = 1$. So, $R_n(x)$ has a cycle of length j if and only if $y^{(n^j-1)(q-1)} = 1$ if and only if $q^2 - 1$, which is the size of the multiplicative group $\mathbb{F}_{q^2}^*$, has a divisor t such that $t \mid (n^j - 1)(q - 1)$ if and only if $q + 1$ has a divisor s where $n^j \equiv 1 \pmod{s}$ and j is the smallest integer with this property. \square

Based on the above theorem and a simple counting technique, we can find the explicit number of cycles of length j for a Rédei function R_n over \mathbb{F}_q .

Theorem 3.14. *The number N_j of cycles of length j of the Rédei function R_n over \mathbb{F}_q with $\gcd(n, q + 1) = 1$ satisfies*

$$jN_j + \sum_{\substack{i \mid j \\ i < j}} iN_i + 1 = \gcd(n^j - 1, q + 1).$$

Proof. Similar to the proof of Theorem 3.13, we look for $y \in \mathbb{F}_{q^2}^*$ such that $y^{(n^j-1)(q-1)} = 1$ and $y^{(n^j-1)} \in \mathbb{F}_q$. Let ρ be a primitive element of $\mathbb{F}_{q^2}^*$. Let us assume that s_0 is a common divisor of $q + 1$ and $n^j - 1$. Every c with the property $\gcd(c, q + 1) = \frac{q+1}{s_0}$ can raise to a cycle of length j

$$\left(y, R_n(y), R_n^{(2)}(y), \dots, R_n^{(j-1)}(y)\right) = \left(\rho^c, R_n(\rho^c), R_n^{(2)}(\rho^c), \dots, R_n^{(j-1)}(\rho^c)\right).$$

On the other hand $\gcd(c, q + 1) = \frac{q+1}{s_0}$ implies that there exists $t_0 \in \mathbb{N}$ such that $c = \frac{q+1}{s_0}t_0$. Hence, we get

$$(\rho^c)^{(n^j-1)(q-1)} = (\rho)^{c(n^j-1)(q-1)} = (\rho)^{\frac{n^j-1}{s_0}(q^2-1)t_0} = (1)^{\frac{n^j-1}{s_0}t_0} = 1,$$

and

$$(\rho^c)^{(n^j-1)} = (\rho)^{c(n^j-1)} = (\rho)^{\frac{n^j-1}{s_0}(q+1)t_0} = (\rho^{q+1})^{\frac{n^j-1}{s_0}t_0} \in \mathbb{F}_q,$$

where the last expression is true because the powers of $q + 1$ of ρ form \mathbb{F}_q in \mathbb{F}_{q^2} . We are interested in the number of $c = \frac{q+1}{s_0}t_0$ such that $\gcd(c, q + 1) = \frac{q+1}{s_0}$. This is the number of t_0 such that $\gcd(t_0, s_0) = 1$ and $t_0 \leq s_0$, which equals $\phi(s_0)$ where $\phi(\cdot)$ denotes Euler's function. Therefore, summing over all $\phi(s_0)$ with $s_0 \mid \gcd(q+1, n^j-1)$ gives the number of elements that contribute to the cycles of length j and all its divisors i . We have

$$1 + \sum_{i \mid j} iN_i = \sum_{s_0 \mid \gcd(q+1, n^j-1)} \phi(s_0) = \gcd(q + 1, n^j - 1).$$

The last equality is derived from the fact that for every n we have $\sum_{d|n} \phi(d) = n$. We note that 1 in the left hand side accounts for the element 0 since $R_n(0) = 0$ gives an extra fixed point and the above counting enumerates non-zero elements $y \in \mathbb{F}_{q^2}$. \square

Now, we are in a position to state our final expression. The following theorem gives a general condition for a Rédei function to have only cycles of length j and 1.

Theorem 3.15. *The Rédei function R_n of \mathbb{F}_q with $\gcd(n, q+1) = 1$ has all its cycles of length j or 1 if and only if for every divisor s of $q+1$ we have $n \equiv 1 \pmod{s}$ or $j = \text{ord}_s(n)$.*

Corollary 3.16. *The Rédei function R_n of \mathbb{F}_q with $\gcd(n, q+1) = 1$ is self-inverse if and only if $n^2 \equiv 1 \pmod{q+1}$.*

Proof. One can insert $j = 2$ in Theorem 3.15 and produce a self-inverse Rédei function. Thus, $R_n(x)$ is a self-inverse function if and only if

$$(12) \quad \text{for all } s|q+1, \quad \begin{cases} n \equiv 1 \pmod{s} \text{ or} \\ n^2 \equiv 1 \pmod{s}. \end{cases}$$

It is easy to see that (12) is equivalent to $n^2 \equiv 1 \pmod{q+1}$ and this simply shows that $m = n$ for $R_m(x) = R_n^{-1}(x)$. \square

Now we are able to find all self-inverse Rédei functions by using the above corollary. Also, one can apply self-inverse Rédei functions to produce self-inverse Rédei interleavers.

Theorem 3.17. *Let $q+1 = p_0^{k_0} p_1^{k_1} \dots p_r^{k_r}$. The permutation of \mathbb{F}_q given by the Rédei function R_n has cycles of the same length j or fixed points if and only if one of the following conditions holds for each $0 \leq \ell \leq r$*

- $n \equiv 1 \pmod{p_\ell^{k_\ell}}$,
- $j = \text{ord}_{p_\ell^{k_\ell}}(n)$ and $j|p_\ell - 1$,
- $j = \text{ord}_{p_\ell^{k_\ell}}(n)$, $k_\ell \geq 2$ and $j = p_\ell$.

Proof. We begin stating some easy lemmas and propositions.

Lemma 3.18. *If $n \equiv b \pmod{p^\ell}$, then $n^p \equiv b^p \pmod{p^{\ell+1}}$ for all $\ell \geq 1$.*

Lemma 3.19. *Let $j = \text{ord}_{p^\ell}(n)$. Then $j = \text{ord}_{p^{\ell+1}}(n)$ or $jp = \text{ord}_{p^{\ell+1}}(n)$.*

Proposition 3.20. *We have that $j = \text{ord}_{p^k}(n)$ and $j|p-1$ if and only if $j = \text{ord}_{p^\ell}(n)$ for all $1 \leq \ell \leq k$.*

Lemma 3.21. *Let $p = \text{ord}_{p^k}(n)$ for some $k \geq 2$. Then either $2 = p = \text{ord}_{p^\ell}(n)$ for $2 \leq \ell \leq k$ or $n \equiv 1 \pmod{p^\ell}$ for $1 \leq \ell \leq k$.*

Lemma 3.22. *Let $j = \text{ord}_s(n)$, $j = \text{ord}_\ell(n)$ and $\gcd(s, \ell) = 1$. Then $j = \text{ord}_{s\ell}(n)$.*

Lemma 3.23. *Let $j = \text{ord}_s(n)$, $n \equiv 1 \pmod{\ell}$ and $\gcd(s, \ell) = 1$. Then $j = \text{ord}_{s\ell}(n)$.*

Now we proceed to give the main proof for Theorem 3.17 (\Leftarrow) If $n \equiv 1 \pmod{p_\ell^{k_\ell}}$ for all $0 \leq \ell \leq r$, then $R_n(x)$ is the identity permutation. Suppose that $1 < j = \text{ord}_{p_\ell^{k_\ell}}(n)$ for some of the ℓ 's and $n \equiv 1 \pmod{p_\ell^{k_\ell}}$ for the others. Proposition 3.20 and Lemma 3.21 guarantee that $j = \text{ord}_{p_\ell^k}(n)$ or $n \equiv 1 \pmod{p_\ell^k}$

for all $0 \leq \ell \leq r$ and $1 \leq k \leq k_\ell$. Now, if $t|(q+1)$, then by Lemmas 3.22 and 3.23, we have that, $j = \text{ord}_t(n)$ or $n \equiv 1 \pmod{t}$. Hence, by Theorem 3.15, all the cycles have length j or 1.

(\implies) Suppose that all the cycles have the same length j . Then, by Theorem 3.15, $j = \text{ord}_t(n)$ or $n \equiv 1 \pmod{t}$ for all t that divides $q+1$. This holds in particular for $t = p_\ell^{k_\ell}$; $0 \leq \ell \leq r$. We only have to prove that, if $j = \text{ord}_{p_\ell^{k_\ell}}(n)$ then $j|(p_\ell - 1)$ or $j = p_\ell$; $k_\ell \geq 2$. Suppose that $1 \neq j = \text{ord}_{p_\ell^{k_\ell}}(n)$. If $k_\ell = 1$ then $j|(p_\ell - 1)$ and we are done. If $k_\ell \geq 2$ and $j \nmid (p_\ell - 1)$, then Proposition 3.20 implies that $j \neq \text{ord}_{p_\ell^k}(n)$ for some $k < k_\ell$. Let s be the largest one such that $j \neq \text{ord}_{p_\ell^s}(n)$. Then $n \equiv 1 \pmod{p_\ell^s}$ because otherwise, by Theorem 3.15, there would be a cycle of length different from j . By Lemma 3.18, $i^{p_\ell} \equiv 1 \pmod{p_\ell^{s+1}}$. But $j = \text{ord}_{p_\ell^{s+1}}(n)$ implies that $j|p_\ell$ and hence $j = p_\ell$. It has to be noted that the above proof is similar to the proof of Theorem 2 in [24]. \square

3.6. SKOLEM INTERLEAVERS. In this section we construct other types of interleavers. In this case our underlying structure are various types of Skolem sequences including k -extended, hooked and (j, n) -generalized. Clearly, we shall use (j, n) -generalized Skolem sequences to produce interleavers with cycles of length 1 and j only. First of all, let us make a slight modification to every type of Skolem sequences to turn them into a consistent form for using in our construction method. Let us assume that $S = (s_1, \dots, s_t)$ be a Skolem sequence (not a generalized one but maybe k -extended or even hooked) over a set of integers D . For every $i \in [n]$ if ℓ , $2 \leq \ell \leq t$, is the largest index such that $s_\ell = i$ then convert s_ℓ to $-s_\ell$ i.e. put $-i$ instead of i in the ℓ th position. In the case that S is a (j, n) -generalized Skolem sequence just insert $-(j-1)i$ instead of i in the ℓ th position. These changed Skolem sequences are called *modified Skolem sequences* (*modified generalized Skolem sequences*, respectively). Now, our strategy is to reorder any set of integers of length t , say $I = \{1, \dots, t\}$, based on a modified (generalized) Skolem sequence of length t , $S = (s_1, \dots, s_t)$.

Let us assume that S^m is a modified (generalized) Skolem sequence of order n with alphabet $[n]$. If $i \in [n]$ repeats on positions u and v where $u < v$, then $s_u^m = i$ and $s_v^m = -i$ and $v - u = i$. Now, define the interleaver Π_S by sending u to v . More precisely

$$(13) \quad \Pi_S(u) = u + s_u^m = u + i = v.$$

We observe that holes or zeros of our sequence produce fixed points. In other words, if $s_h^m = 0$ for some $1 \leq h \leq m$, then $\Pi_S(h) = h + s_h^m = h + 0 = h$. If we imagine the indices of our modified Skolem sequence as time and the amounts of our Skolem sequence as location (domain), then this interleaver may be interpreted as a combination of time and domain.

Example 3.24. The sequence $S = (2, 5, 2, 6, 1, 1, 5, 3, 4, 6, 3, 0, 4)$ is a hooked Skolem sequence. First we have to convert it to a modified hooked Skolem sequence. If we denote the modified version of S by S^m , we get

$$S^m = (2, 5, -2, 6, 1, -1, -5, 3, 4, -6, -3, 0, -4).$$

Thus, based on (13), we have

$$\begin{aligned} \Pi_S(1) &= 1 + s_1^m = 3, & \Pi_S(2) &= 2 + s_2^m = 7, & \Pi_S(3) &= 3 + s_3^m = 1, \\ \Pi_S(4) &= 4 + s_4^m = 10, & \Pi_S(5) &= 5 + s_5^m = 6, & \Pi_S(6) &= 6 + s_6^m = 5, \end{aligned}$$

$$\begin{aligned}\Pi_S(7) &= 7 + s_7^m = 2, & \Pi_S(8) &= 8 + s_8^m = 11, & \Pi_S(9) &= 9 + s_9^m = 13, \\ \Pi_S(10) &= 10 + s_{10}^m = 4, & \Pi_S(11) &= 11 + s_{11}^m = 8, & \Pi_S(12) &= 12 + s_{12}^m = 12, \\ \Pi_S(13) &= 13 + s_{13}^m = 9.\end{aligned}$$

The above equalities induce the following Skolem interleaver

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 3 & 7 & 1 & 10 & 6 & 5 & 2 & 11 & 13 & 4 & 8 & 12 & 9 \end{pmatrix}.$$

Theorem 3.25. *Let Π_S be an interleaver constructed using a modified Skolem sequence. Then Π_S is a self-inverse interleaver. Furthermore, if we use a modified (j, n) -generalized Skolem sequence, then Π_S has only cycles of length j or 1.*

Proof. Let $S = (s_1, \dots, s_m)$ and its modified version, S^m , are used to produce Π_S . For every $i \in D$ there exist indices u and v such that $u < v$, $s_u^m = i$, $s_v^m = -i$ and $v - u = i$. Based on the definition of Π_S we get

$$\Pi_S(u) = u + s_u^m = u + i = v, \quad \Pi_S(v) = v + s_v^m = v - i = u,$$

and also if $s_h^m = 0$ for some h we have $\Pi_S(h) = h + s_h^m = h + 0 = h$. Hence, Π_S is a self-inverse interleaver. On the other hand let us suppose that S is a (j, n) -generalized Skolem sequence and S^m be its modified version. For every $i \in [n]$ there are indices $r_1, r_2 = r_1 + i, \dots, r_j = r_1 + (j-1)i$ such that $s_{r_1}^m = s_{r_2}^m = \dots = s_{r_{j-1}}^m = i$ and $s_{r_j}^m = -(j-1)i$. Since $r_{w+1} = r_w + i$ and based on (13) we get

$$\Pi_S(r_w) = r_w + s_{r_w}^m = r_w + i = r_{w+1}$$

for $1 \leq w \leq j-1$, and for $w = j$ we have

$$\Pi_S(r_j) = r_j + s_{r_j}^m = r_j - (j-1)i = r_1.$$

It means that (r_1, r_2, \dots, r_j) builds a cycle of length j for Π_S . More explicitly, $\Pi_S(r_w) = r_{w+1}$ for $1 \leq w \leq j-1$ and $\Pi_S(r_j) = r_1$. What we did for holes (zeros) of a modified Skolem sequence, can be likewise done for generalized Skolem sequences. \square

4. CONCLUSION AND FURTHER RESEARCH

Some deterministic interleavers have been introduced and investigated based on permutation functions over finite fields. Well-known permutation functions have been explained. Rédei functions treated in detail. We derived an exact formula for the inverse of every Rédei function. We gave the cycle structure of these functions and provided the exact number of cycles of a certain length j . Specifically, we focused on self-inverse permutation functions which can produce self-inverse interleavers and have the potential to reduce the memory consumption [26].

The Skolem interleavers could be useful for making interleavers with a known cycle structure. We have the same possible option and characteristics for Dickson and Rédei permutations as well. More specifically, suppose that we are given a cycle structure (i_1, \dots, i_n) for a permutation σ where i_j denotes the number of cycles of length j in σ . Clearly, we have $\sum_{j=1}^n j i_j = n$. Assume that we want to construct an interleaver following the structure of the permutation σ . We need i_j cycles of length j . For example we can use a $(j, j i_j)$ -generalized Skolem sequence. Even monomial, Dickson or Rédei permutations can be employed to produce these cycles. To this end, the cycle structure of monomial, Dickson and Rédei permutations can be determined using results of Theorems 3.4, 3.7 and 3.15.

Clearly, more theoretical studies to provide maximum achievable minimum distance, spread factor and dispersion [10, 11, 34] of these interleavers are of interest. Moreover, finding a suitable relationship between cycles and sequences introduced in [33] and cycle structure of permutations seems worthwhile.

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